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Extensions of the Khinchine-Wisser Theorem

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§1. Introduction:

As an outgrowth of considerations related to the "Poincaré recurrence theorem" Khinchine proved:

If $A_1, A_2, \ldots, A_n, \ldots$ is a given infinite sequence of measurable sets such that

$$(1.1) \qquad P(A_n) \geq \alpha > 0, \qquad \text{for all } n;$$

and the sequence is "stationary" , i.e.

$$(1.2) \qquad P(A_r\, A_s) = P(A_i\, A_j), \qquad \text{for } r-s = i-j;$$

then given any $\varepsilon > 0$, there exists an infinite subsequence A_{i_k}, $k = 1,2,\ldots$, such that

$$(1.3) \qquad P(A_{i_k}\, A_{i_\ell}) > (1-\varepsilon)\alpha^2 \, . \; *$$

Wisser provided a much simplified proof of this, and at the same time dropped the assumption of stationarity. In this note various extensions of Wisser's result are obtained,

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* Note that the theorem is trivially true for $\varepsilon \geq 1$. Furthermore by choosing the A_n as independent sets such that $P(A_n) \longrightarrow \alpha$ as $n \longrightarrow \infty$, we see that the assertion (1.3) is in a sense best possible.

which focus on providing subsequences, of given infinite sequences, on which the probability of any finite intersection is bounded from below in a "natural way" .

§2. Notations.

In order to facilitate the statements and proofs which are to be presented it is convenient to utilize various notations, which are listed below.

For $\mathcal{S}$ any sequence of sets $A_1, A_2, \ldots$, (possibly a finite sequence), we set

$$\text{(a)}\ \underline{\mathcal{S}}(n) = \{A_1, A_2, \ldots, A_n\}, \text{ for each integer } n \geq 1;$$

and

$$\text{(b)}\ \bar{\mathcal{S}}(n) = \{A_{n+1}, \ldots, \}, \text{ for each integer } n \geq 1.$$

If $\mathcal{T}$ is a subsequence of $\mathcal{S}$ we write $\mathcal{T} \subset \mathcal{S}$, or $\mathcal{S} \supset \mathcal{T}$; and in the special case where $\mathcal{T}$ is a finite sequence we denote by $\mathcal{T} \cup \mathcal{S}$ the sequence formed by first listing $\mathcal{T}$ and then following it with $\mathcal{S}$. Thus, for example, for each integer $n \geq 1$

$$\mathcal{S} = \underline{\mathcal{S}}(n) \cup \bar{\mathcal{S}}(n);$$

$$\underline{\mathcal{S}}(n) \subset \mathcal{S}$$

and

$$\bar{\mathcal{S}}(n) \subset \mathcal{S}.$$

In addition:

(c) $\mathcal{S}^k = \left\{ A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k} \; ; \; A_{i_\mu} \in \mathcal{S}, \; i_\mu \neq i_\nu \text{ for } \mu \neq \nu \right\}$,

for each integer $k \geq 1$. Note then that $\mathcal{S}^1 = \mathcal{S}$.

(d) $[\mathcal{S}]$ = the collection of all sets which occur in some $\mathcal{S}^k$, $k \geq 1$.

(e) For a set $B \in [\mathcal{S}]$ define $k = \rho(B)$ to be the smallest integer $k \geq 1$ such that $B \in \mathcal{S}^k$.

(f) $\mathcal{S}$ is called "(n, β) linked" if for every pair of sets A, B such that $B \in [\underline{\mathcal{S}}(n)]$, $A \in \bar{\mathcal{S}}(n)$, we have $P(B) > 0$, and

$$P_B(A) \geq \beta P(A) ,$$

where $P_B(A)$ denotes the conditional probability of A, assuming B has occurred.

(g) $\mathcal{S}$ is called "completely β linked" if it is (n,β) linked for every integer $n \geq 1$.

(h) Define $\quad \triangle(\mathcal{S}) = \underset{A \in \mathcal{S}}{\text{g.l.b.}} \; P(A)$

§3. <u>A generalization of the Khinchine - Wisser Theorem.</u> In terms of the notation provided in the previous section one possible extension of the Khinchine - Wisser Theorem may be given as follows:

<u>Theorem 3.1</u> Given an infinite sequence $\mathcal{S}$ such that $\Delta(\mathcal{S}) > 0$,* and any $\varepsilon > 0$, there exists a subsequence $\mathcal{S}' \subset \mathcal{S}$, such that $\mathcal{S}'$ is completely $(1-\varepsilon)$ linked.

<u>Remark:</u> It is clear that one need only consider the case where $\varepsilon < 1$.

The theorem given above is obtained by means of an inductive construction, whose relation to the theorem is provided in the following lemma.

<u>Lemma 3.1</u> Given an infinite sequence of subsequences of $\mathcal{S}$,

$$\mathcal{S} \supset \mathcal{S}'_1 \supset \mathcal{S}'_2 \supset \cdots \supset \mathcal{S}'_n \supset \mathcal{S}'_{n+1} \supset \cdots$$

such that

(3.1) $\mathcal{S}_n'$ is (n, λ_n) linked, $\lambda_n > 1-\varepsilon$;

(3.2) $\underline{\mathcal{S}}_n'(n) \subset \underline{\mathcal{S}}'_{n+1}(n+1)$,

for all $n \geq 1$; define

(3.3) $\mathcal{S}' = \lim\limits_{n \to \infty} \underline{\mathcal{S}}_n'(n)$.

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* The condition $\Delta(\mathcal{S}) > 0$ can be weakened to $\underline{\lim}\, P(A_n) > 0$.

Then, the sequence $\mathcal{S}'$ is completely 1-ϵ linked.

__Proof:__ Suppose that $B \ \varepsilon \ [\mathcal{S}'(n)]$, $A \ \varepsilon \ \bar{\mathcal{S}}'(n)$. Then $B \ \varepsilon \ [\mathcal{S}'_n(n)]$ and $A \ \varepsilon \ \bar{\mathcal{S}}_n(n)$. Then, since $\mathcal{S}'_n$ is (n, λ_n) linked, $P(B) > 0$, and

$$P_B(A) \geq \lambda_n \ P(A) > (1-\varepsilon) \ P(A) \ .$$

Hence, $\mathcal{S}'$ is $(n, 1-\varepsilon)$ linked for every $n \geq 1$, and consequently completely $(1-\varepsilon)$ linked.

Thus in order to prove Theorem 3.1 we need only construct the sequence of subsequences described in Lemma 3.1. This construction will in turn be made to depend on the following lemma.

__Lemma 3.2.__ Given an infinite sequence $\mathcal{T} = (A_1, A_2, \ldots)$ such that $\triangle(\mathcal{T}) > 0$, and a set B such that $P(B) > 0$ and

$$(3.4) \quad P_B(A_n) \geq (1-\lambda) \ P(A_n) \ ,$$

for all $n \geq 1$; for any $\eta > 0$ (however small) there exists an infinite subsequence $\mathcal{T}' = (A_{i_1}, \ldots) \subset \mathcal{T}$ such that

(3.5) $\quad P_{BA_{i_1}}(A_{i_\mu}) \geq (1-\lambda-\eta)\, P(A_{i_\mu}),$

for all $\mu \geq 1$.

Proof: Note first that for $\mu = 1$, (3.5) holds automatically, so that once the subsequence $\mathcal{T}'$ is constructed we need only focus on the verification of (3.5) for $\mu \geq 2$.

Negating the assertion of the lemma provides that for each $A_i \in \mathcal{T}'$,

$$P_{BA_i}(A_j) < (1-\lambda-\eta)\, \rho(A_j)$$

for all $j > i$, except for a finite number of exceptions. This in turn allows us to construct an infinite subsequence $\mathcal{T}''$ of $\mathcal{S}$, $\mathcal{T}'' = \{A_{j1}, A_{j2}, \dots\}$, such that

(3.6) $$P_{BA_{j_\sigma}}(A_{j_\tau}) < (1-\lambda-\eta)\, P(A_{j_\tau}),$$

for all pairs (σ, τ) with $\sigma < \tau$.

Letting $\chi(\omega/A)$ denote the characteristic function of the set A, we have

$$0 \le \int_B \left(\sum_{\sigma=1}^{n} \chi(\omega | BA_{j_\sigma}) - \frac{1}{P(B)} \sum_{\sigma=1}^{n} P(BA_{j_\sigma}) \right)^2 dP(\omega),$$

$$(3.7) \quad \sum_{\substack{\sigma=1,\ldots,n \\ \tau=1,\ldots,n}} P(BA_{j_\sigma} A_{j_\tau}) \ge \frac{1}{P(B)} \left\{ \sum_{\sigma=1}^{n} P(BA_{j_\sigma}) \right\}^2 .$$

We next rewrite (3.7) in the form

$$(3.8) \quad \sum_{\sigma=1}^{n} P(BA_{j_\sigma}) \sum_{\sigma<\tau\le n} P_{BA_{j_\sigma}}(A_{j_\tau})$$

$$\ge \sum_{\sigma=1}^{n} P(BA_{j_\sigma}) \sum_{\sigma<\tau\le n} P_B(A_{j_\tau}) + O(n).$$

Using (3.4) on the right of (3.8), we get

$$(3.9) \quad \sum_{\sigma=1} P(BA_{j_\sigma}) \sum_{\sigma<\tau\le n} P_{BA_{j_\sigma}}(A_{j_\tau})$$

$$\ge (1-\lambda) \sum_{\sigma=1}^{n} \sum_{\sigma<\tau\le n} P(BA_{j_\sigma}) P(A_{j_\tau}) + O(n).$$

Using (3.6) on the left of (3.9) this yields in turn

$$(3.10) \quad 1-\lambda-\eta \ge 1-\lambda + O\left(n \Big/ \sum_{\substack{\sigma,\tau\le n \\ \sigma<\tau}} P(BA_{j_\sigma}) P(A_{j_\tau}) \right).$$

However, since by (3.4) and the definition of $\triangle(\overline{\mathcal{J}}) > 0$,

$$\sum_{\substack{\sigma,\tau\le n \\ \sigma<\tau}} P(BA_{j_\sigma}) P(A_{j_\tau}) \ge (1-\lambda) \triangle^2(\mathcal{J}) \frac{n(n-1}{2} P(B),$$

we see that by taking n sufficiently large (3.10) implies $\eta \leq 0$, which is a contradiction. The lemma then follows.

Proof of Theorem 3.1

By applying the Lemma 3.2 we now proceed to the construction of the subsequences $\mathcal{S}'_n$ described in Lemma 3.1. We begin by applying Lemma 3.2 with

$$\mathcal{T} = \mathcal{S},$$

$$B = \text{the whole space},$$

$$\lambda = 0,$$

$$\eta = \varepsilon/2 .$$

The subsequence $\mathcal{T}'$ provided by Lemma 3.2 is then taken as $\mathcal{S}'_1$, and by (3.5) this is clearly $(1, \lambda_1)$ linked with $\lambda_1 = 1-\varepsilon/2$.

Now assume that $\mathcal{S} \supset \mathcal{S}_1' \supset \mathcal{S}_2' \supset \ldots \supset \mathcal{S}'_n$, $(n > 1)$, have been constructed so as to satisfy (3.1) and (3.2), with $\lambda_k = 1-\varepsilon \sum_{i=1}^{k} 1/2^i$. Consider, then, a $B \varepsilon [\mathcal{S}'_n(n)]$. Since $\mathcal{S}'_n$ is (n,λ_n) linked, $P(B) > 0$, and for any $A \varepsilon \mathcal{S}'_n(n)$ we have

$$P_B(A) \geq \lambda_n P(A).$$

Thus $\mathcal{T} = \mathcal{S}'_n(n)$ satisfies (3.4) with $\lambda = 1-\lambda_n$, and $\triangle\left(\mathcal{S}_n'(n)\right) \geq \triangle(\mathcal{S}) > 0$, so that Lemma 3.2 may be applied

with $\eta = \varepsilon/2^{n+1}$ This provides a subsequence $\mathcal{T}'_1 \subset \mathcal{S}_n(n)$, on which (3.5) holds. Proceeding next with another B in $[\underline{\mathcal{S}}'_n(n)]$ we extract a subsequence $\mathcal{T}'_2 \subset \mathcal{T}'_1$ such that (3.5) holds, this time with the new B. Repeating this process until the finite number of B in $[\underline{\mathcal{S}}'_n(n)]$ are all accounted for, we arrive at a $\mathcal{T}'_m$ which is an infinite subsequence of $\mathcal{S}'_n(n)$, having the property that for all B ε $[\underline{\mathcal{S}}'_n(n)]$, and any $A_{i_\mu} \varepsilon \mathcal{T}'_m$, $\mu \geq 1$,

$$(3.11) \quad P_{BA_{i_1}}(A_{i_\mu}) \geq (1-\varepsilon \sum_{i=1}^{n+1} 1/2^i)\, P(A_{i_\mu}) .$$

We then define the infinite sequence $\mathcal{S}'_{n+1}$ by

$$(3.12) \qquad \mathcal{S}'_{n+1} = \left\{ \underline{\mathcal{S}}'_n(n) \, , \, \mathcal{T}'_m \right\}.$$

Clearly, (3.2) holds, and (3.11) together with the facts that $\mathcal{S}'_n$ is (n,λ_n) linked, and $\mathcal{T}'_m \subset \mathcal{S}'_n(n)$, implies that $\mathcal{S}'_{n+1}$ is $(n+1,\lambda_{n+1})$ linked, (where $\lambda_{n+1} = 1-\varepsilon \sum_{i=1}^{n+1} 1/2^i$). Thus we see that $\mathcal{S}'_{n+1} \subset \mathcal{S}'_n$ and satisfies the requirements of (3.1) and (3.2). The inductive character of the construction of the $\mathcal{S}'_n$ has been established, and Theorem 3.1 is proved.

<u>Theorem 3.2</u> Given an infinite sequence $\mathcal{S}$ such that $\Delta(\mathcal{S}) > 0$, and any ε, $0 > \varepsilon > 1$, there exists a subsequence $\mathcal{S}' \subset \mathcal{S}$, such that for any $E_i \in \mathcal{S}'$, $i = 1, \ldots, k$, (any) $k \geq 1$)

$$(3.13) \qquad P\left(\bigcap_{i=1}^{k} E_i\right) \geq (1-\varepsilon)^{k-1} \prod_{i=1}^{k} P(E_i)$$

Proof: The subsequence $\mathcal{S}'$ is taken here to be the one provided by Theorem 3.1. We further assume that the E_i are listed here in order of this occurence in the sequence $\mathcal{S}'$; and set

$$B_j = \bigcap_{i=1}^{k} E_i \ , \quad j = 1, \ldots, k-1 \ .$$

then

$$(3.14) \qquad P\left(\bigcap_{i=1}^{k} E_i\right) = P(E_1) \prod_{j=1}^{k-1} P_{B_j}(E_{j+1}) \ ;$$

and since $\mathcal{S}'$ is completely $(i-\varepsilon)$ linked

$$(3.15) \quad P_{B_j}(E_{j+1}) \geq (1-\varepsilon) \, P(E_{j+1}) \ , \quad j = 1, \ldots, k-1 \ .$$

(3.13) is then an immediate consequence of (3.14) and (3.15).

<u>Corollary</u>. The subsequence $\mathcal{S}'$ produced in Theorem 3.2 has the property that for any $E \in [\mathcal{S}']$,

$$(3.16) \qquad P(E) \geq \left\{ (1-\varepsilon)\, \Delta(\mathscr{S}) \right\}^{\rho(E)} .$$

Proof: This is immediate from (3.13) and the fact that $\Delta(\mathscr{S}') \geq \Delta(\mathscr{S})$.

The above corollary is in form most closely related to the Khinchine - Wisser theorem, which only requires condition (3.16) for the case when $\rho(E) = 2$.

3a. <u>A counterexample</u>. At first glance, the above results suggest the possibility of something like (3.13) without the full hypothesis $\Delta(\mathscr{S}) > 0$. In particular, it is natural to attempt to replace the hypothesis $\Delta(\mathscr{S}) > 0$ by $\sum P(A_i) = \infty$, where the A_i are the sets of the sequence $\mathscr{S}$. The following simple counterexample shows that this is not possible.

Let $B_1, B_2, \ldots$ be any infinite sequence of mutually disjoint sets such that $P(B_i) > 0$, for all i . We then form the sequence $\mathscr{S}$ by first repeating B_1, n_1 times, followed by B_2 repeated n_2 times, etc. We then have

$$\sum_{\mathscr{S}} P(A_i) = \sum_{i=1}^{\infty} n_i \, P(B_i) ,$$

which diverges, and as rapidly as we like, by an appropriate choice of the n_i. Furthermore, it is clear that (3.13) does not hold on any infinite subsequence of $\mathscr{S}$, for any $k > 1$.

§4. Further generalizations of the Khinchine-Wisser Theorem

The generalization of the Khinchine-Wisser Theorem provided by Theorem 3.1 focuses on estimating certain conditional probabilities from below. However, as we've seen, the estimate (3.13) given in Theorem 3.2 is, in form, more closely related to the original results of Khinchine and Wisser. In considering the question of possible refinements of this estimate, it is only natural to ask whether or not the subsequence $\mathcal{S}'$ can be chosen so that the dependence on k occuring in the factor $(1-\varepsilon)^k$, in (3.13), is removed. The answer to this question is in the affirmative, and we formulate this formally as:

Theorem 4.1. Given an infinite sequence $\mathcal{S}$ such that $\Delta(\mathcal{S}) > 0$, and any ε, $0 < \varepsilon < 1$, there exists an infinite subsequence $\mathcal{S}' \subset \mathcal{S}$, such that for any $E_i \in \mathcal{S}'$, $i = 1,\ldots,k$, (any $k \geq 1$), we have

$$(4.1) \qquad P\left(\bigcap_{i=1}^{k} E_i\right) \geq (1-\varepsilon) \prod_{i=1}^{k} P(E_i) .$$

It is clear that, by the same derivation used to obtain Theorem 3.2 from Theorem 3.1, Theorem 4.1 would be an easy consequence of the following.

Theorem 4.2 Given an infinite sequence $\mathcal{S}$ such that $\Delta(\mathcal{S}) > 0$, and any infinite sequence of real numbers ε_i, $i = 1,2,\ldots$, $0 < \varepsilon_i < 1$, there exists an infinite subsequence $\mathcal{S}' \subset \mathcal{S}$ such that for

$$A_{i_1}, A_{i_2}, \ldots, A_{i_{k-1}}, A_{i_k} \varepsilon \mathcal{S}'$$

(any $k \geq 1$), and any $A \varepsilon \mathcal{S}'$ which appears in the sequence after all the A_{i_μ}, $\mu = 1,\ldots k$, we have

$$(4.2) \qquad P_{A_{i_1} \cap \ldots \cap A_{i_k}}(A) \geq (1- \varepsilon_k)P(A).$$

We note that Theorem 3.1 is simply the special case of Theorem 4.2 in which all the $\varepsilon_i = \varepsilon$. The added strength of Theorem 4.2 lies in the fact that ε_i may be chosen so as to tend to zero as $i \longrightarrow \infty$, and as rapidly as we please.

We shall in fact obstain the following slightly stronger assertion:

Theorem 4.2A. Given an infinite sequence $\mathcal{S}$ such that $\Delta(\mathcal{S}) > 0$, and any function $\phi(u) > 0$, such that $\phi(u) \longrightarrow 0$ as $u \longrightarrow \infty$, there exists an infinite subsequence $\mathcal{S}' \subset \mathcal{S}$ such that for all

$$A_{i_1}, A_{i_2}, \ldots, A_{i_{k-1}}, A_{i_k} \in \mathcal{S}',$$

(any $k \geq 1$), where the A_{i_μ} are listed in order of occurence in $\mathcal{S}'$, we have

$$(4.2A) \qquad P_{A_{i_1} \cap \ldots \cap A_{i_{k-1}}}(A_k) \geq (1-\phi(i_k))P(A_k).$$

Remarks.

(1) Theorem 4.2A provides a corresponding strengthening of Theorem 4.1 in which the assertion (4.1) is replaced by

$$(4.1A) \quad P\left(\bigcap_{\mu=1}^{k} A_{i_\mu}\right) \geq \prod_{\mu=2}^{k} (1-\phi(i_\mu))P(A_{i_\mu})$$

(2) Theorem 4.2A is an immediate consequence of the apparently weaker statement which asserts that there exists <u>some</u> function $\phi_o(u) > 0$, $\lim_{u \to \infty} \phi_o(u) = 0$, and a subsequence $\mathcal{S}'$ such that (4.2A) holds, (with ϕ_o for ϕ). This follows, since by a further extraction of a subsequence $\mathcal{S}''$ from $\mathcal{S}'$, it can be arranged that ϕ_o goes to zero, as fast as we like, over the original $\mathcal{S}'$ indices of the sets of $\mathcal{S}''$. That is, if ϕ is given, $\mathcal{S}'' = (E_1, E_2, \ldots)$, $E_\mu = A_{i_\mu}$ in $\mathcal{S}'$, we arrange that

$$\phi_o(i_\mu) \leq \phi(\mu).$$

The remainder of this section will be devoted to the proof of Theorem 4.2A. For this purpose, it is useful to introduce the function $\varepsilon(B|\mathcal{S})$, defined for every measurable set B, by

$$\varepsilon(B|\mathcal{S}) = \varliminf_{A\varepsilon\mathcal{S}} \left\{ 1- \frac{P_B(A)}{P(A)} \right\} \tag{4.3}$$

This may also be written as

$$\varlimsup_{A\varepsilon\mathcal{S}} \frac{P_B(A)}{P(A)} = 1- \varepsilon(B|\mathcal{S}). \tag{4.4}$$

<u>Lemma 4.1</u> Given an infinite sequence $\mathcal{S}$ there exists an infinite subsequence $\mathcal{S}' \subset \mathcal{S}$ such that for all $B \in [\mathcal{S}']$, we have

$$\lim_{A\varepsilon\mathcal{S}'} \frac{P_B(A)}{P(A)} = 1-\varepsilon(B|\mathcal{S}'). \tag{4.5}$$

Proof: Let A_1 be the first element of $\mathcal{S}$. Choose a subsequence $\mathcal{S}^{(11)}$ of $\mathcal{S}$ such that

$$\lim_{A\varepsilon\mathcal{S}^{(11)}} \frac{P_{A_1}(A)}{P(A)} = 1-\varepsilon(A_1|\mathcal{S}^{(11)}).$$

Let A_2 be the first element of $\mathcal{S}^{(11)}$, (after A_1), and choose an infinite subsequence $\mathcal{S}^{(12)} \subset \mathcal{S}^{(11)}$ such that

$$\lim_{A \in \mathcal{S}^{(12)}} \frac{P_{A_2}(A)}{P(A)} = 1 - \varepsilon(A_2 | \mathcal{S}^{(12)}).$$

Note that we also automatically have

$$\lim_{A \in \mathcal{S}^{(12)}} \frac{P_{A_1}(A)}{P(A)} = 1 - \varepsilon(A_1 | \mathcal{S}^{(12)});$$

and in fact $\varepsilon(A_1 | \mathcal{S}^{(11)}) = \varepsilon(A_1 | \mathcal{S}^{(11)})$. Continuing with this process we produce an infinite sequence

$$\mathcal{S}^{\langle 1 \rangle} = \{A_1, A_2, \ldots\}$$

such that for any A_i, $i = 1, 2, \ldots$

$$\lim_{A \in \mathcal{S}^{\langle 1 \rangle}} \frac{P_{A_i}(A)}{P(A)} = 1 - \varepsilon(A_i | \mathcal{S}^{\langle 1 \rangle}).$$

Next, we fix A_1, A_2 and extract an infinite subsequence $\mathcal{S}^{(21)} \subset \mathcal{S}^{\langle 1 \rangle}\mathcal{S}$, such that

$$\lim_{A \in \mathcal{S}^{(21)}} \frac{P_{A_1 \cap A_2}(A)}{P(A)} = 1 - \varepsilon(A_1 \cap A_2 | \mathcal{S}^{(21)}).$$

Let A_3 (this requires a convenient renaming of sets) be the first set of $\mathcal{S}^{(21)}$, (after A_1 or A_2), and choose a subsequence $\mathcal{S}^{(22)} \subset \mathcal{S}^{(21)}$ such that

$$\lim_{A\in\mathcal{S}^{(21)}} \frac{P_{A_1\cap A_2}(A)}{P(A)} = 1 - \varepsilon(A_1\cap A_2|\mathcal{S}^{(21)}).$$

Let A_3 (this requires a convenient renaming of sets) be the first set of $\mathcal{S}^{(21)}$, (after A_1 or A_2), and choose a subsequence $\mathcal{S}^{(22)} \subset \mathcal{S}^{(21)}$ such that

$$\lim_{A\in\mathcal{S}^{(22)}} \frac{P_{A_1\cap A_3}(A)}{P(A)} = 1 - \varepsilon(A_1\cap A_3|\mathcal{S}^{(22)})$$

and

$$\lim_{A\in\mathcal{S}^{(22)}} \frac{P_{A_2\cap A_3}(A)}{P(A)} = 1 - \varepsilon(A_2\cap A_3|\mathcal{S}^{(22)}).$$

Clearly, we also have

$$\lim_{A\in\mathcal{S}^{(22)}} \frac{P_{A_1\cap A_2}(A)}{P(A)} = 1-\varepsilon(A_1\cap A_2|\mathcal{S}^{(22)}).$$

Letting A_4 (again a renaming) be the first set in $\mathcal{S}^{(22)}$ (after A_1, A_2, A_3), and continuing this process we produce a subsequence $\mathcal{S}^{<2>} \subset \mathcal{S}^{<1>}$ such that

$$\lim_{A\in\mathcal{S}^{<2>}} \frac{P_B(A)}{P(A)} = 1 - \varepsilon(B|\mathcal{S}^{<2>})$$

for are $B \in [\mathcal{S}^{<2>}]$ such that $\rho(B) \leq 2$. We note further that $\mathcal{S}^{<2>}$ has retained the first two elements of $\mathcal{S}^{<1>}$. Continuing then analogously we can extract an infinite subsequence $\mathcal{S}^{<3>} \subset \mathcal{S}^{<2>}$ such that $\mathcal{S}^{<3>}$ retains the first three elements of $\mathcal{S}^{<2>}$ and such that

$$\lim_{A \in \mathcal{S}^{<3>}} \frac{P_B(A)}{P(A)} = 1 - \varepsilon(B | \mathcal{S}^{<3>})$$

for all $B \in \mathcal{S}^{<3>}$ such that $\rho(B) \leq 3$. Continuing indirectively in this way we define $\mathcal{S}^{<k>}$ for all $k \geq 1$ with analogous properties, so that

$$\mathcal{S}' = \lim \underline{\mathcal{S}}^{<k>}(k)$$

satisfies (4.5), for all $B \in [\mathcal{S}']$.

We note that the sequence $\mathcal{S}'$ produced by the above lemma has the property that for $\mathcal{S}''$ any infinite subsequence of $\mathcal{S}'$, and all $B \in [\mathcal{S}']$,

$$\varepsilon(B | \mathcal{S}') = \varepsilon(B, \mathcal{S}'') \tag{4.6}$$

and

$$\lim_{A \in \mathcal{S}''} \frac{P_B(A)}{P(A)} = 1 - \varepsilon(B | \mathcal{S}''). \tag{4.7}$$

In other words, the characteristic properties which dictated the construction of $\mathcal{S}'$ are inherited by every infinite

subsequence of $\mathcal{S}'$.

This observation leads to the following sharpening of Lemma 4.1.

Lemma 4.2. Let $\mathcal{S}$ be a given infinite sequence of sets. For any sequence $\mathcal{T}$ of sets, and any $A \varepsilon \mathcal{T}$, let $r(A|\mathcal{T})$ denote the order of occurrence of A in $\mathcal{T}$. There exists an infinite subsequence $\mathcal{S}' \subset \mathcal{S}$, such that for any $B \varepsilon[\mathcal{S}']$,

$$(4.8) \qquad \frac{P_B(A)}{P(A)} = 1 - \varepsilon(B|\mathcal{S}') + E(B,A|\mathcal{S}')$$

with

$$(4.9) \qquad |E(B,A|\mathcal{S}')| \leq r(A|\mathcal{S}')^{-3}$$

for all are $A \varepsilon \mathcal{S}'$ such that $r(A|\mathcal{S}')$ is larger than the order of occurrence of any "factor" of B.

Proof: We begin by extracting from $\mathcal{S}$ the infinite subsequence provided by Lemma 4.1, such that (4.5) holds, which we here denote by $\mathcal{S}^*$. From the remarks preceding this lemma we see that if $\mathcal{T}$ is any infinite subsequence of $\mathcal{S}^*$, and $B \varepsilon[\mathcal{S}^*]$, $A \varepsilon \mathcal{T}$,

$$(4.10) \qquad E(B,A,\mathcal{S}^*) = E(B,A,\mathcal{T}).$$

Thus since

$$r(A|\mathcal{S}') \leq r(A|\mathcal{S}^*)$$

we see from (4.10) that if (4.9) were to hold on $\mathcal{S}^*$ it would also certainly hold on $\mathcal{S}'$. More generally, if (4.9) holds on any infinite subsequence of $\mathcal{S}^*$, its validity is maintained under further extraction of subsequences. using this fact, together with an inductive construction similar to the one used in proving Lemma 4.1, it is an easy matter to extract the desired subsequence $\mathcal{S}'$ out of the subsequence $\mathcal{S}^*$.

From this point on, we shall adopt the following convention: Given any sequence $\mathcal{S}$, let $\mathcal{S}^*$ be the subsequence of $\mathcal{S}$ provided by Lemma 4.2. Then for any B ε [$\mathcal{S}^*$] we write

$$\varepsilon(B) = \varepsilon(B|\mathcal{S}^*). \tag{4.11}$$

Thus ε(B) is defined invariantly with respect to all infinite subsequence of $\mathcal{S}^*$, which eliminates the necessity of carrying such sequences notationally in the function ε(B).

Lemma 4.3. Let $\mathcal{S}$ be a given infinite sequence of sets such that $\triangle(\mathcal{S}) > 0$. Then there exists an infinite subsequence $\mathcal{S}' \subset \mathcal{S}^*$ such that for all B ε[$\mathcal{S}'$], and B = Ω (the whole space), either

(a) $\varepsilon(BA) < 0$ for <u>all</u> $A \in \mathcal{S}'$ which appear in $\mathcal{S}'$ after all of the factors of B, or

(b) $\varepsilon(BA \geq 0$ for <u>all</u> $A \in \mathcal{S}'$ which appear in $\mathcal{S}'$ after all the factors of B.

Furthermore,

$$(4.12) \qquad \overline{\sum_{\substack{A \in \mathcal{S} \\ \varepsilon(A)>0}}}{}' \ \varepsilon(A) < \infty,$$

and

$$(4.13) \qquad \overline{\lim_{A \in \mathcal{S}}}{}' \ \varepsilon(BA \leq \varepsilon(B), \quad \text{for all } B \in [\mathcal{S}'].$$

A $B \in [\mathcal{S}']$ such that (a) holds will be referred to as of <u>"type ν"</u> ; and if (b) holds, of "<u>type</u> π". The subsequence $\mathcal{S}'$ will thus have the further property that <u>if</u>

$$B = B'B'', \; B', B'' \in [\mathcal{S}'],$$

such that all factors of B'' follow all those of B', in $\mathcal{S}'$; and B' is of type ν, <u>then</u> B is of type ν. Thus, it follows that if $B \in [\mathcal{S}]$ is of type π , then for any decomposition $B = B'B''$ such as described above, B' must also be of type π.

Proof: (§) We begin by extracting the subsequence $\mathcal{J}^*$ and further thinning it out if necessary so that (3.13) holds with $\varepsilon = 1/2$ for all ≥ 1.. then, either

$(a)_1$ $\varepsilon(A) < 0$ for infinitely many $A \varepsilon \mathcal{J}^*$

or

$(b)_1$ $\varepsilon(A) \geq 0$ for all but a finite number of $A \varepsilon \mathcal{J}^*$.

If alternative $(a)_1$ holds, let $\mathcal{J}^{(1)}$ denote the infinite subsequence consisting of the $A \varepsilon \mathcal{J}^*$ such that $\varepsilon(A) < 0$. If alternative $(b)_1$ holds, let $\mathcal{J}^{(1)}$ denote the infinite subsequence of $\mathcal{J}^*$ obtained by deleting the finite number of $A \varepsilon \mathcal{J}^*$ such that $\varepsilon(A) < 0$. Thus, either

$(a)_1$ $\varepsilon(A) < 0$ for all $A \varepsilon \mathcal{J}^{(1)}$

or

$(b)_1$ $\varepsilon(A) \geq 0$ for all $A \varepsilon \mathcal{J}^{(1)}$.

Under alternative $(b)_1$ we next wish to show that

(§ For the convenience of a later argument we assume to begin with that (by a subsequence extraction) it is arranged once and for all that

$$\lim_{A \varepsilon \mathcal{J}} P(A) = \alpha > 0.$$

$$(4.14) \qquad \sum_{A \in \mathcal{A}^{(1)}} \varepsilon(A) < \infty .$$

To achieve this we consider the inequality derived in the proof of Lemma 3.2. If $\mathcal{A}^{(1)} = \{A_1, A_2, \ldots\}$ this gives

$$(4.15) \qquad \sum_{\substack{i,j \\ 1 \leq i \leq n \\ 1 \leq j \leq n}} P(A_i A_j) \geq \sum_{\substack{i,j \\ 1 \leq i \leq n \\ 1 \leq j \leq n}} P(A_i)P(A_j)$$

From the construction of $\mathcal{A}^{(1)}$ as a subsequence of $\mathcal{A}^*$, we have for $i > j$,

$$P(A_i A_j) = P(A_i)P(A_j)\left[(1-\varepsilon(A_j)) + O\left(\frac{1}{i^3}\right)\right]$$

so that (4.15) yields

$$O(n) + \sum_{\substack{n \geq i > j \geq 1 \\ i,j}} P(A_i)P(A_j)\left[(1-\varepsilon(A_j)) + O\left(\frac{1}{i^3}\right)\right]$$

$$\geq \sum_{\substack{i,j \\ n \geq i > j > 1}} P(A_i)P(A_j) .$$

This in turn reduces to

$$\sum_{\substack{n \geq i > j \geq 1 \\ i,j}} \varepsilon(A_j)P(A_i)P(A_j) \leq \sum_{\substack{i \leq n \\ j \leq n}} O\left(\frac{1}{i^3}\right) + O(n) = O(n).$$

Since $P(A_i) \geq \triangle = \triangle(\mathcal{S}) > 0$, this gives

$$\sum_{j=1}^{n} \varepsilon(A_j)(n-j) \leq O(n).$$

Finally, since the $\varepsilon(A_j) \geq 0$ for all j, we set

$$\sum_{j=1}^{n/2} \varepsilon(A_j) = O(1),$$

which implies (4.12).*

Next, let A_1 denote the first set in $\mathcal{S}^{(1)}$. Then

<u>either</u>

$(a)_{21}$ $\varepsilon(A_1A) < 0$ for infinitely many $A \varepsilon \mathcal{S}^{(1)}$ which follow A_1;

or

$(b)_{21}$ $\varepsilon(A_1A) \geq 0$ for all but a finite number of $A \varepsilon \mathcal{S}^{(1)}$.

If alternative $(a)_{21}$ holds let $\mathcal{S}^{(21)}$ be the infinite subsequence of $\mathcal{S}^{(1)}$ composed of the infinitely many $A \varepsilon \mathcal{S}^{(1)}$ such that $\varepsilon(A_1A) < 0$, together with A_1 itself. If alternative $(b)_{21}$ holds, let $\mathcal{S}^{(21)}$ consist of A_1 together with those $A \varepsilon \mathcal{S}^{(1)}$ such that $\varepsilon(A_1A) \geq 0$. Thus, either

$(a)_{21}$: $\varepsilon(A_1A) < 0$ for all $A \in \mathcal{S}^{(21)}$ which follow A_1;

<u>or</u>

$(b)_{21}$: $\varepsilon\,(A_1A) \geq 0$ for all $A \in \mathcal{S}^{(21)}$ which follow A_1.

Next we show that if $\mathcal{S}^{(21)}$ was formed under alternative $(a)_{11}$, then alternative $(a)_{21}$ <u>must</u> hold; or equivalently that alternative $(b)_{21}$ <u>cannot</u> hold. This is an immediate consequence of (4.13) in the case $\rho(B) = 1$, which may be proved as follows. Let $\mathcal{S}^{(21)} = \{A_1, A_2, \ldots\}$; then

$$(4.16)\quad P(A_1) \sum_{\substack{i,j \\ \sqrt[3]{n}<i\leq n \\ \sqrt[3]{n}<j\leq n}} P(A_1A_iA_j) \geq \sum_{\substack{i,j \\ \sqrt[3]{n}<i\leq n \\ \sqrt[3]{n}<j\leq n}} P(A_1A_i)P(A_1A_j).$$

Furthermore, our assumptions provide that for $j > i$

$$(4.17)\quad P(A_1A_iA_j) = P(A_j)P(A_1A_i)\left[1 - \varepsilon(A_1A_i) + O\left(\frac{1}{j^3}\right)\right]$$

and for all $j > 1$

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* For the convenience of a later argument, we discard the finite number of $A \in \mathcal{S}^*$ such that

$$\varepsilon(A) \geq 1/2.$$

$$(4.18) \qquad P(A_l A_j) = P(A_l)P(A_j)\left\{1 + O\left(\frac{1}{j^3}\right) - \varepsilon(A_l)\right\}.$$

Inserting (4.18) properly into (4.16) yields

$$P(A_l) \sum_{\substack{i,j \\ \sqrt[3]{n}<i<n \\ \sqrt{n}<j\leq n \\ i<j}} P(A_l A_i A_j)$$

$$\geq P(A_l) \sum_{\substack{i,j \\ \sqrt[3]{n}<i\leq n \\ \sqrt[3]{n}<j\leq n \\ i<j}} P(A_l A_i)P(A_j)\left\{1+O\left(\frac{1}{j^3}\right) - \varepsilon(A_l)\right\} + O(n),$$

or

$$(4.19) \qquad \sum_{\substack{i,j \\ \sqrt[3]{n}<i<j\leq n}} P(A_l A_i A_j) \geq (1-\varepsilon(A_l)) \sum_{\substack{i,j \\ \sqrt[3]{n}<i<j\leq n}} P(A_l A_i)P(A_j) + O(n).$$

But (4.19) implies the existence of a pair $i_o = i_o(n)$, $j_o = j_o(n)$

such that

$$\sqrt[3]{n} < i_o < j_o \leq n,$$

and

$$(4.20) \qquad P(A_l A_{i_o} A_{j_o}) \geq (1-\varepsilon(A_l))\, P(A_l A_{i_o})P(A_{j_o}) + O\left(\frac{1}{n}\right)$$

Combining (4.20) and (4.7), and using that $j_o^3 > n$

$$(4.21) \qquad P(A_{j_o})P(A_1A_{i_o})(-\varepsilon(A_1A_i)$$

$$\geq -\varepsilon(A_1)P(A_1A_{i_o})P(A_{j_o}) + O(\tfrac{1}{n}).$$

Then since $P(A_1A_{j_o})P(A_{j_o})$ is bounded away from zero uniformly in i_o, j_o; [§]

$$0 \geq -\varepsilon(A_1A_{i_o}) \geq -\varepsilon(A_i) + O(\tfrac{1}{n}),$$

This in particular, completes the proof of the fact that if alternative $(a)_1$ holds, alternative $(a)_{21}$ must hold.

Next let A_2 be the first set in $\mathcal{S}^{(21)}$ after A_1.

Then, <u>either</u>

$(a)_{22}$: $\varepsilon(A_2A) < 0$ for infinitely many $A \in \mathcal{S}^{(21)}$ which follow A_2;

or

$(b)_{22}$; $\varepsilon(A_2A) \geq 0$ for all but a finite number of $A \in \mathcal{S}^{(21)}$.

(§) We have this because of our initial arrangement that (3.13) hold with $\varepsilon = 1/2$.

If alternative $(a)_{22}$ holds, let $\mathcal{S}^{(22)}$ be the infinite subsequence of $\mathcal{S}^{(21)}$ composed of the infinitely many $A \varepsilon \mathcal{S}^{(1)}$ such that $\varepsilon(A_2A) < 0$, together with A_1 and A_2. If alternative $(b)_{22}$ holds, let $\mathcal{S}^{(22)}$ consist of A_1, A_2, together with those $A \varepsilon \mathcal{S}^{(21)}$, which follow A_2, and are such that $\varepsilon(A_2A) \geq 0$. Thus either

$(a)_{22}$: $\varepsilon(A_2A) < 0$ for all $A \varepsilon \mathcal{S}^{(22)}$ which follow A_2;

or

$(b)_{22}$: $\varepsilon(A_2A) \geq 0$ for all $A \varepsilon \mathcal{S}^{(22)}$ which follow A_2.

By argument entirely analogous to the one given above for $\mathcal{S}^{(21)}$, it can be shown that (4.13) holds for $\rho(B) = 2$ and hence that alternative $(a)_{22}$ must occur if alternative $(a)_1$ occurred.

Repeating the above procedure inductively produces an infinite sequence of subsequences $\mathcal{S}^{(2,k)}, k = 1,2,\ldots,$ such that

$(i)_k$ $\quad \mathcal{S}^{(2,(k+1)} \subset \mathcal{S}^{(2,k)}$

$(ii)_k$ $\quad \mathcal{S}^{(2,k+1)}(k+1 \supset \underline{\mathcal{S}}^{(2,k)}(k)$

$(iii)_k$ $\quad$ for any $A_* \varepsilon \underline{\mathcal{S}}^{(2,k)}(k)$, either

$(a)_{2,k}$: $\varepsilon(A_*A) < 0$ for all $A \varepsilon \mathcal{S}^{(2,k)}$ which follow A_* ;

or

$(b)_{2,k}$: $\varepsilon(A_*A) \geq 0$ for all $A \varepsilon \mathcal{S}^{(2,k)}$ which follow A_*

$(iv)_k$ the alternative $(a)_{2,k}$ holds if the alternative $(a)_1$ held originally.

We then see that the infinite sequence

$$(4.21) \qquad \mathcal{S}^{(2)} = \lim_{k \to \infty} \underline{\mathcal{S}}^{(2,k)}(k)$$

is a subsequence of $\mathcal{S}^{(1)}$, such that

$$(4.22) \qquad \underline{\mathcal{S}}^{(1)}(1) \simeq \underline{\mathcal{S}}^{(2)}(2) \;;$$

and for any $B \varepsilon [\mathcal{S}^{(2)}]$ such that $\rho(B) \leq 1$ <u>either</u>

$(a)_2$: $\varepsilon(BA) < 0$ for all $A \varepsilon \mathcal{S}^{(2)}$ which follow all factors of B

<u>or</u>

$(b)_2$: $\varepsilon(BA) \geq 0$ for all $A \varepsilon \mathcal{S}^{(2)}$ which follow all factors of B.

Furthermore, alternative $(a)_2$ holds if $(a)_1$ held originally.

Next we take the first two sets A_1, A_2 in $\mathcal{S}^{(2)}$ and for $B = A_1 \cap A_2$, we have that either

$(a)_{312}$: $\varepsilon(BA) < 0$ for infinitely many $A \in \mathcal{S}^{(2)}$ which follow A_2;

<u>or</u>

$(b)_{312}$: $\varepsilon(BA) \geq 0$ for all but a finite number of sets $A \in \mathcal{S}^{(2)}$.

If alternative $(a)_{312}$ holds, let $\mathcal{S}^{(312)}$ denote the infinite subsequence of $\mathcal{S}^{(2)}$ consisting of A_1, A_2, and those $A \in \mathcal{S}^{(2)}$ following A_2, such that $\varepsilon(BA) < 0$. If alternative $(b)_{312}$ holds, let $\mathcal{S}^{(312)}$ denote the infinite subsequence of $\mathcal{S}^{(2)}$ consisting of A_1, A_2, together with those $A \in \mathcal{S}^{(2)}$ following A_2, such that $\varepsilon(BA) \geq 0$. Thus, either

$(a)_{312}$: $\varepsilon(BA) < 0$ for all $A \in \mathcal{S}^{(312)}$ which follow A_2;

<u>or</u>

$(b)_{312}$: $\varepsilon(BA) \geq 0$ for all $A \in \mathcal{S}^{(312)}$ which follow A_2.

then by exactly the same argument as used in treating the alternative $(a)_{21}$, (just replace the A_1 there by B), it follows that if $(a)_2$ held originally for $B = A_1$ alternative $(a)_{312}$ must hold.

Letting A_3 be the first set of $\mathcal{S}^{(312)}$ after A_2, we process $B = A_1 \cap A_3$, and $B = A_2 \cap A_3$ successively to produce

a subsequence $\mathcal{S}^{(3,1-3)}$ of $\mathcal{S}^{(312)}$ such that A_1, A_2, A_3 are its first three sets and for $B = A_1 \cap A_2$, or $A_2 \cap A_3$, or $A_1 \cap A_3$, either

$(a)_{3,1-3}$: $\varepsilon(BA) < 0$ for all $A \in \mathcal{S}^{(3,1-3)}$ which follow A_3;

<u>or</u>

$(b)_{3,1-3}$: $\varepsilon(BA) \geq 0$ for all $A \in \mathcal{S}^{(3,1-3)}$ which follow A_3.

The alternative which holds depends on B, but again if for example $(a)_2$ held originally for $B = A_1$, $(a)_{3,1-3}$ must hold for $B = A_1 \cap A_2$ and $B = A_1 \cap A_3$.

Continuing then with this inductive construction one produces finally an infinite sequence $\mathcal{S}^{(3)} \subset \mathcal{S}^{(2)}$ such that

$$\underline{\mathcal{S}}^{(2)}(2) \subset \underline{\mathcal{S}}^{(3)}(3) ,$$

and then for each $B \in [\mathcal{S}^{(3)}]$ such that $\rho(B) \leq 2$, either

$(a)_3$: $\varepsilon(BA) < 0$ for all $A \in \mathcal{S}^{(3)}$ which follow all the factors of B;

<u>or</u>

$(b)_3$: $\varepsilon(BA) \geq 0$ for all $A \in \mathcal{S}^{(3)}$ which follow all the factors of B.

Furthermore, if $B = A'A''$, (or A'), (where A'' follows A'), $A', A'' \in \mathcal{S}^{(3)}$, where A' satisfies $(a)_2$, as a B, (or A' satisfies $(a)_1$ as a B) then alternative $(a)_3$ must hold.

Finally then these inductive constructions produce an infinite sequence of subsequences $\mathcal{S}^{(k)}$, $k = 1,2,\ldots$, such that:

(4.23) $$\mathcal{S}^{(k)} \subset \mathcal{S}^{(k+1)};$$

(4.24) $$\underline{\mathcal{S}}^{(k)}(k) \subset \underline{\mathcal{S}}^{(k+1)}(k+1) ;$$

and for each $B \in [\mathcal{S}^{(k)}]$ such that $\rho(B) \leq k-1$, either

$(a)_k$: $\varepsilon(BA) < 0$ for all $A \in \mathcal{S}^{(k)}$ which follow all the factors of B;

<u>or</u>

$(b)_k$: $\varepsilon(BA) \geq 0$ for all $A \in \mathcal{S}^{(k)}$ which follow all the factors of B.

Furthermore if $B = B'B''$, $B', B'' \in [\mathcal{S}^{(k)}]$, where $\rho(B) \leq k-1$, all factors of B'' follow all the factors of B' in $\mathcal{S}^{(k)}$, and B' satisfied alternative $(a)_t$ for some $t < k$; then B must satisfy alternative $(a)_k$.

From the above it follows readily that

$$\mathcal{S}' = \lim \underset{\sim}{\mathcal{S}}^{(k)}(k),$$

is an infinite subsequence of $\mathcal{S}^*$ possessing the properties quoted in the lemma.

Remarks

It follows from Lemma 4.3 that if the sequence $\mathcal{S}^*$ contains infinitely many A such that $\varepsilon(A) < 0$, then there is an infinite subsequence $\mathcal{S}' \subset \mathcal{S}^*$ such that for all $B \varepsilon [\mathcal{S}']$,

(4.25) $$\varepsilon(B) < 0 .$$

It is not clear that this conclusion may also be asserted if we assume only that the original sequence $\mathcal{S}$ (with $\triangle(\mathcal{S}) > 0$) contains infinitely many A such that

$$\varepsilon(A|\mathcal{S}) < 0,$$

and possibly this is false.

In any event, if $\varepsilon(A) < 0$ infinitely often in $\mathcal{S}^*$, (4.25), together with (4.8), and (4.9), yields Theorem 4.2A immediately. Hence, in the following we need only consider the alternative wherein $\varepsilon(A) \geq 0$ for all $A \varepsilon \mathcal{S}''$. By a further subsequence extraction we can arrange that either

(i) $\varepsilon(A) = 0$ for all $A \varepsilon \mathcal{S}'$ ($\mathcal{S}'$ an infinite sequence);

or

(ii) $\varepsilon(A) > 0$ for all $A \varepsilon \mathcal{S}'$ ($\mathcal{S}'$ an infinite sequence).

under case (ii), we have from (4.12) that

$$\sum_{A \varepsilon \mathcal{S}'} \varepsilon(A) < \infty .$$

The following lemmas will provide, in particular, that alternative (ii) is impossible.

Lemma 4.4 Let $\mathcal{S}'$ be the infinite sequence of Lemma 4.3, and assume $\varepsilon(A) \geq 0$ for all $A \varepsilon \mathcal{S}'$. Let $\chi(\omega|A)$ denote the characteristic function of the set A; and let $\mathcal{F}$ denote the set of fifth powers of the positive integers. Furthermore, if $\mathcal{S}' = \{A_1, A_2, \ldots\}$, let M be the set of ω such that

$$(4.26) \qquad \left| \sum_{i=1}^{n} \chi(\omega|A_i) - \sum_{i=1}^{n} P(A_i) \right| \geq n^{5/8}$$

for infinitely many $n \varepsilon \mathcal{F}$. Then

$$(4.27) \qquad P(M) = 0 .$$

Proof: we have

$$\int \left(\sum_{i=1}^{n} \chi(\omega | A_i) - \sum_{i=1}^{n} P(A_i) \right)^2 dP$$

$$= \sum_{i,j \leq n} P(A_i A_j) - \sum_{i,j \leq n} P(A_i)P(A_j)$$

$$= \sum_{i<j\leq n} \left(- \varepsilon(A_i) + O(\frac{1}{j^3}) \right) P(A_i)P(A_j) + O(n)$$

$$\leq O(n) .$$

Thus, if M_n denotes the set of ω such that (4.26) holds, this implies

$$(4.27) \qquad P(M_n) = O(\frac{1}{n^{1/4}}) .$$

(4.27) in turn implies that

$$\sum_{n\varepsilon\mathcal{F}} P(M_n) < \infty,$$

so that by the Borel-Cantelli lemma, $M = \{ M_n \text{ i.o., } n \varepsilon \mathcal{F} \}$ has probability zero, and the lemma is proved.

<u>Lemma 4.5</u> Let $\mathcal{E}'$ be the infinite sequence as described in Lemma 4.4. Let $B \varepsilon [\mathcal{A}']$, then

$$(4.28) \qquad \varliminf_{n\in\mathcal{F}} \frac{\sum_{i\le n} P(BA_i)}{\sum_{i\le n} P(A_i)} \ge P(B).$$

Proof: Applying Fatou's lemma, we have

$$\varliminf_{n\in\mathcal{F}} \int_B \frac{\sum_{i\le n} \chi(\omega|A_i)}{\sum_{i\le n} P(A_i)}\, dP \ge \varliminf_{n\in\mathcal{F}} \int_{B\cap M} \frac{\sum_{i\le n} \chi(\omega|A_i)}{\sum_{i\le n} P(A_i)}\, dP$$

(4.29)

$$\ge \int_{B\cap \bar{M}} \varliminf_{n\in\mathcal{F}} \frac{\sum_{i\le n} \chi(\omega|A_i)}{\sum_{i\ge n} P(A_i)}\, dP.$$

Clearly,

$$(4.30) \qquad \int_B \frac{\sum_{i\le n} \chi(\omega|A_i)}{\sum_{i\le n} P(A_i)}\, dP = \frac{\sum_{i\le n} P(BA_i)}{\sum_{i\le n} P(A_i)}.$$

Also, by Lemma 4.4, on $\bar{M}$ (the complement of M)

$$\lim_{n\varepsilon\mathcal{F}} \frac{\sum_{i\geq n} \chi(\omega|A_i)}{\sum_{i\leq n} P(A_i)} = 1,$$

so that

$$(4.31) \qquad \int_{B\cap\bar{M}} \lim_{n\varepsilon\mathcal{F}} \frac{\sum_{i\leq n} \chi(\omega|A_i)}{\sum_{i\leq n} P(A_i)} = P(B\cap\bar{M}) = P(B),$$

since $P(\bar{M}) = 1$. From (4.29), (4.30), and (4.31), (4.28) follows.

Lemma 4.6. Set $\mathscr{S}'$ be the infinite sequence as described in Lemma 4.4. Let $B \varepsilon [\mathscr{S}']$, then

$$(4.32) \qquad \varepsilon(B) \leq 0.$$

Thus as a consequence of (4.8) and (4.9),

$$(4.33) \qquad P_B(A) \geq \left\{1 - \frac{1}{[r(A|\mathscr{S}')]^3}\right\} P(A).$$

for all $A \varepsilon \mathscr{S}'$ such that A follows all factors of B in $\mathscr{S}'$.

Proof: Suppose that for a given $B \varepsilon [\mathscr{S}']$, the A_i with $i \leq k = k(B)$, occur in $\mathscr{S}''$ after all the factors of B. Since

$$\underline{\lim}_{n\in\mathcal{T}} \frac{\sum_{k\le i\le n} P(BA_i)}{\sum_{k\le i\le n} P(A_i)} = \underline{\lim}_{n\in\mathcal{T}} \frac{\sum_{i\le n} P(BA_i)}{\sum_{i\le n} P(A_i)}$$

we obtain from (4.28) that

$$\underline{\lim}_{n\in\mathcal{T}} \frac{\sum_{k\le i\le n} P(BA_i)}{\sum_{k\le i\le n} P(A_i)} \ge P(B) \;. \tag{4.34}$$

On the other hand, from (4.8) and (4.9),

$$\sum_{k\le i\le n} P(BA_i) \le \sum_{k\le i\le n} P(B)P(A_i)\ (1-\varepsilon(B)) + O(\frac{1}{i^3}),$$

which together with (4.34) yields (4.32).

Finally, the assertion (4.33) of the above lemma, taken together with the second remark which follows the statement of Theorem 4.2A, completes the proof of Theorem 4.2A.

A bit more may be extracted from the above arguments. if in the proof of Lemma 4.5 the Lebesgue convergence Theorem is applied instead of Fatou's lemma, one obtains

$$\underline{\lim}_{\substack{n\to\infty \\ n\in\mathcal{T}}} \frac{\sum_{i\le n} P(BA_i)}{\sum_{i\le n} P(A_i)} = P(B). \tag{4.35}$$

Then, as in the proof of Lemma 4.6, this leads to

$$(4.36) \qquad \varepsilon(B) = 0,$$

for all $B \varepsilon [\mathscr{S}']$. This result, which is obtained in the case where $\varepsilon(A) = 0$ in $\mathscr{S}'$, may then be combined with Lemma 4.3 to give

<u>Theorem 4.2B.</u> Given an infinite sequence $\mathscr{S}$, with $\triangle(\mathscr{S}) > 0$, <u>either</u> (I) there exists an infinite subsequence $\mathscr{S}'' \subset \mathscr{S}$ such that

$$(4.37) \qquad \varepsilon(B) < 0,$$

for all $B \varepsilon [\mathscr{S}'']$; <u>or</u>

(II) there exists an infinite subsequence $\mathscr{S}' \subset \mathscr{S}$ such that

$$(4.38) \qquad \varepsilon(B) = 0,$$

for all $B \varepsilon [\mathscr{S}']$.

§5. <u>One More Counterexample.</u> The form of the assertion (4.2A) suggests the conjecture that there must exist a subsequence of $\mathscr{S}$ ($\triangle(\mathscr{S}) > 0$) on which (4.1A) may be strengthened to

$$(5.1) \qquad P\left(\bigcap_{\mu=1}^{k} A_{i_\mu}\right) \geq \left(1 - \phi(k)\right) \prod_{\mu=1}^{k} P(A_{i_\mu})$$

where $\phi(k) > 0$, is some function such that $\lim_{k \to \infty} \phi(k) = 0$. we will now show that this conjecture is not true.

Note first that if $\mathcal{S}'$ is the subsequence of $\mathcal{S}$ on which (5.1) holds, (5.1) also holds relative to any subsequence of $\mathcal{S}'$. We then choose the subsequence $\mathcal{S}''$, provided by Theorem 4.2B, on which one of the alternatives (I) or (II) must hold. If for $A_i, A_j \in \mathcal{S}''$, $i \neq j$,

$$(5.2) \qquad P(A_iA_j) < P(A_i)P(A_j),$$

then it is alternative (II) which must hold. Assume this to be the case. Then it follows that for any fixed A_{i_1}, A_{i_2} in $\mathcal{S}''$, we may choose

$$A_{i_3(k)} \cdots A_{i_k(k)} \in \mathcal{S}'',$$

so that $i_\nu(k) \to \infty$ as $k \to \infty$ in such a way as to provide

$$(5.3) \qquad P\left(A_{i_1} A_{i_2} A_{i_3(k)} \cdots A_{i_k(k)}\right) \sim \left(1+E(A_{i_1},A_{i_2})\right) P(A_{i_1})P(A_{i_2}) \prod_{\mu=3}^{k} P(A_{i_\mu(k)}) .$$

But then (5.1) and (5.3) imply that

$$1 + E\left(A_{i_1}, A_{i_2}\right) \geq 1 - \phi(k) + o(1)$$

as $k \to \infty$, which in turn implies

$$E\left(A_{i_1}, A_{i_2}\right) \geq 0.$$

This, however, implies that

$$P(A_{i_1} A_{i_2}) \geq P(A_{i_1})P(A_{i_2})$$

in contradiction to (5.2).

Thus (5.1) is impossible if (5.2) holds, and the proposed conjecture is proved false once we produce a sequence on which (5.2) holds. We construct such a sequence, inductively, as follows. Let A_1 be any set such that $P(A_1)$ is greater than 3/4; i.e. $P(A_1) = 1 - \varepsilon_1$, $\varepsilon_1 < 1/4$, and $P(A_1) < 1$, i.e. $\varepsilon_1 > 0$. Assume then that the A_j, $j < n$ have been constructed with $P(A_j) = 1 - \varepsilon_j$, $0 < \varepsilon_j \leq \frac{1}{2^{j+1}}$.

Since

$$P\left(\bigcap_{j=1}^{n-1} A_j\right) = 1 - P\left(\bigcup_{j=1}^{n-1} \bar{A}_j\right)$$

$$\geq 1 - \sum_{j=1}^{n-1} \varepsilon_j > 0,$$

We can choose for A_n any set such that

$$P(A_n) = 1-\varepsilon_n, \ \bar{A}_n \subset \bigcap_{j=1}^{n-1} A_j \ ;$$

and where

$$0 < \varepsilon_n \leq \min \left(\frac{1}{2^{n+1}} , P\left(\bigcap_{j=1}^{n-1} A_j \right) \right) .$$

Then for $1 \leq j < n$, we have

$$P(A_j A_n) = P(A_j) - P(A_j \bar{A}_n) = P(A_j) - \varepsilon_n$$

and

$$P(A_j)P(A_n) = (1-\varepsilon_n)P(A_j) \quad .$$

Since

$$(1-\varepsilon_n)P(A_j) > P(A_j) - \varepsilon_n \ ,$$

it follows that

$$P(A_j A_n) < P(A_j)P(A_n) \ .$$

Thus the inductive nature of the construction is established, and we obtain a sequence such that (5.2) holds.

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